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(Anti)symmetric multivariate exponential functions and corresponding Fourier transforms

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Abstract
We define and study symmetrized and antisymmetrized multivariate exponential functions. They are defined as determinants and antideterminants of matrices whose entries are exponential functions of one variable. These functions are eigenfunctions of the Laplace operator on the corresponding fundamental domains satisfying certain boundary conditions. To symmetric and antisymmetric multivariate exponential functions there correspond Fourier transforms. There are three types of such Fourier transforms: expansions into the corresponding Fourier series, integral Fourier transforms and multivariate finite Fourier transforms. Eigenfunctions of the integral Fourier transforms are found.

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1. Introduction

In mathematical and theoretical physics, very often we deal with functions on the Euclidean space E_n which are symmetric or antisymmetric with respect to the permutation (symmetric) group S_n. For example, such functions describe collections of identical particles. Symmetric and antisymmetric solutions appear in the theory of integrable systems. Characters of finite-dimensional representations of semisimple Lie algebras are symmetric functions. Moreover, according to the Weyl formula for these characters, each such character is a ratio of antisymmetric functions.

The aim of this paper is to describe and study symmetrized and antisymmetrized multivariate exponential functions and the corresponding Fourier transforms. Antisymmetric multivariate exponential functions (we denote them by $E^-_\lambda(x)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $x = (x_1, x_2, \ldots, x_n)$) are determinants of $n \times n$ matrices, whose entries are the usual exponential functions of one variable, $E^-_\lambda(x) = \det(\mathrm{e}^{2\pi \mathrm{i}\lambda_i x_j})^n_{i,j=1}$. The symmetric multivariate exponential functions $E^+_\lambda(x)$ are antideterminants of the same $n \times n$ matrices (for a definition of antideterminants see below).

As in the case of the exponential functions of one variable, we may consider three types of antisymmetric and symmetric multivariate exponential functions.

(a) Functions $E^-_m(x)$ and $E^+_m(x)$ with $m = (m_1, m_2, \ldots, m_n)$, $m_i \in \mathbb{Z}$; they determine Fourier series expansions in multivariate symmetric and antisymmetric exponential functions.

(b) Functions $E^-_\lambda(x)$ and $E^+_\lambda(x)$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{R}$; these functions determine integral multivariate Fourier transforms.

(c) Functions $E^-_\lambda(x)$ and $E^+_\lambda(x)$, where $x = (x_1, x_2, \ldots, x_n)$ take a finite set of values; they determine multivariate finite Fourier transforms.

Functions (b) are antisymmetric (symmetric) with respect to the elements of the permutation group S_n. (Anti)symmetries of functions (a) are described by a wider group, since exponential functions $e^{2\pi \mathrm{i} mx}$, $m \in \mathbb{Z}$, of one variable are invariant with respect to shifts $x \to x + k, k \in \mathbb{Z}$. These (anti)symmetries are described by the elements of the affine symmetric group S_n^{aff} which is a product of the group S_n and the group T_n, consisting of shifts in the space E_n by vectors $r = (r_1, r_2, \ldots, r_n)$, $r_j \in \mathbb{Z}$. A fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$ of the group S_n^{aff} is a certain bounded subset of $\mathbb{R}^n$ (see subsection 2.2).

The functions $E^+_\lambda(x)$ give solutions of the Neumann boundary value problem on a closure of the fundamental domain $F(S_n)$. The functions $E^-_\lambda(x)$ are solutions of the Laplace equation $\Delta f = \mu f$ on the domain $F(S_n)$ vanishing on the boundary $\partial F(S_n)$ of $F(S_n)$.

Functions on the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$ can be expanded into series in the functions (a). These expansions are an analogue of the usual Fourier series for functions of one variable. Functions (b) determine an (anti)symmetrized Fourier integral transform on the fundamental domain $F(S_n)$ of the symmetric group S_n. This domain consists of points $x \in E_n$, such that $x_1 > x_2 > \cdots > x_n$.

Functions (c) are used to determine (anti)symmetric finite (that is, on a finite set) Fourier transforms. These finite Fourier transforms are given on grids consisting of points in the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$.

Symmetric and antisymmetric exponential functions are closely related to symmetric and antisymmetric orbit functions defined in [1, 2] and studied in detail in [3, 4]. In fact, symmetric and antisymmetric exponential functions are connected with orbit functions corresponding to the Coxeter–Dynkin diagram A_n. Discrete orbit function transforms, corresponding to Coxeter–Dynkin diagrams of low order, were studied in detail and it was shown that they are very useful for applications [5–13].

The exposition of the theory of orbit functions in [3, 4] strongly depends on the theory of Weyl groups, properties of root systems, etc. In this paper we avoid this dependence. We use only the permutation (symmetric) group and properties of determinants and antideterminants. It is well known that a determinant $\det(a_{ij})^n_{i,j=1}$ of the $n \times n$ matrix $(a_{ij})^n_{i,j=1}$ is defined as

$$\begin{aligned}\det(a_{ij})^n_{i,j=1} &= \sum_{w \in S_n} (\det w) a_{1,w(1)} a_{2,w(2)} \cdots a_{n,w(n)} \\ &= \sum_{w \in S_n} (\det w) a_{w(1),1} a_{w(2),2} \cdots a_{w(n),n}, \end{aligned} \tag{1}$$

where S_n is the symmetric group of n symbols $1, 2, \ldots, n$, the set $(w(1), w(2), \ldots, w(n))$ means the set $w(1, 2, \ldots, n)$ and $\det w$ denotes a determinant of the transform w, that is, $\det w = 1$ if w is an even permutation and $\det w = -1$ otherwise. Along with a determinant, we shall use an antideterminant $\det^+$ of the matrix $(a_{ij})_{i,j=1}^n$ which is defined as the sum of all terms, entering to the expression for the corresponding determinant, taken with the sign +,

$$\det{}^+(a_{ij})_{i,j=1}^n = \sum_{w\in S_n} a_{1,w(1)}a_{2,w(2)}\cdots a_{n,w(n)} = \sum_{w\in S_n} a_{w(1),1}a_{w(2),2}\cdots a_{w(n),n}. \tag{2}$$

Symmetrized and antisymmetrized multivariate polynomials were studied by several authors (see, for example, [14, 15]). In this paper, we investigate symmetric and antisymmetric multivariate exponential functions.

2. Symmetric and antisymmetric multivariate exponential functions

A symmetric multivariate exponential function of $x = (x_1, x_2, \ldots, x_n)$ is defined as the function

$$\begin{aligned} E^+_\lambda(x) &\equiv E^+_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x) = \det{}^+(\mathrm{e}^{2\pi\,\mathrm{i}\lambda_i x_j})_{i,j=1}^n \\ &= \det{}^+\begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_1} & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_2} & \ldots & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_n} \\ \mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_1} & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_2} & \ldots & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_n} \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_1} & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_2} & \ldots & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_n} \end{pmatrix} \\ &\equiv \sum_{w\in S_n} \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_{w(1)}}\,\mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_{w(2)}}\cdots\mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_{w(n)}} = \sum_{w\in S_n}\mathrm{e}^{2\pi\,\mathrm{i}\langle\lambda, wx\rangle}, \end{aligned} \tag{3}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a set of real numbers, which determines the function $E^+_\lambda(x)$ and $\langle\lambda, x\rangle$ denotes the scalar product in the n-dimensional Euclidean space E_n, $\langle\lambda, x\rangle = \sum_{i=1}^n \lambda_i x_i$. When $\lambda_1, \lambda_2, \ldots, \lambda_n$ are integers, we denote this set of numbers as $m \equiv (m_1, m_2, \ldots, m_n)$,

$$E^+_{m_1,m_2,\ldots,m_n}(x) = \det{}^+(\mathrm{e}^{2\pi\,\mathrm{i}m_i x_j})_{i,j=1}^n. \tag{4}$$

It is seen from expression (2) for an antideterminant $\det^+$ that the symmetric exponential functions $E^+_\lambda(x)$ satisfy the relation

$$E^+_\lambda(x_1 + a, x_2 + a, \ldots, x_n + a) = \mathrm{e}^{2\pi\,\mathrm{i}(\lambda_1+\lambda_2+\cdots+\lambda_n)a}E^+_\lambda(x). \tag{5}$$

This means that it is enough to consider the function $E^+_\lambda(x)$ on the hyperplane

$$x_1 + x_2 + \cdots + x_n = b,$$

where b is a fixed number (we denote this hyperplane by $\mathcal{H}_b$). A transition from one hyperplane $\mathcal{H}_b$ to another $\mathcal{H}_c$ is fulfilled by multiplication by a usual exponential function $\mathrm{e}^{2\pi\,\mathrm{i}|\lambda|(c-b)}$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$,

$$E^+_\lambda(x)|_{x\in\mathcal{H}_b} = \mathrm{e}^{2\pi\,\mathrm{i}(\lambda_1+\lambda_2+\cdots+\lambda_n)(c-b)}E^+_\lambda(x)|_{x\in\mathcal{H}_c}.$$

It is useful to consider the functions $E^+_\lambda(x)$ on the hyperplane $\mathcal{H}_0$. For $x \in \mathcal{H}_0$ we have the relation

$$E^+_{\lambda_1+\nu,\lambda_2+\nu,\ldots,\lambda_n+\nu}(x) = E^+_\lambda(x). \tag{6}$$

It is seen from expression (2) for an antideterminant $\det^+$ that its expression does not change under permutations of rows or under permutations of columns. This means that for any permutation $w \in S_n$ we have

$$E^+_\lambda(wx) = E^+_\lambda(x), \qquad E^+_{w\lambda}(x) = E^+_\lambda(x). \tag{7}$$

Therefore, it is enough to consider only symmetric exponential functions $E^+_\lambda(x)$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n.$$

Such λ are called *dominant*. The set of all dominant λ is denoted by D_+. Below, considering symmetric exponential functions $E_\lambda^+(x)$, we assume that $\lambda \in D_+$.

Antisymmetric multivariate exponential functions of $x = (x_1, x_2, \ldots, x_n)$ are defined as the functions

$$E_\lambda^-(x) \equiv E_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}^-(x) := \det(\mathrm{e}^{2\pi\,\mathrm{i}\lambda_i x_j})_{i,j=1}^n$$
$$\equiv \det\begin{pmatrix} \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_1} & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_2} & \cdots & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_n} \\ \mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_1} & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_2} & \cdots & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_1} & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_2} & \cdots & \mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_n} \end{pmatrix}$$
$$\equiv \sum_{w\in S_n} (\det w)\, \mathrm{e}^{2\pi\,\mathrm{i}\lambda_1 x_{w(1)}}\, \mathrm{e}^{2\pi\,\mathrm{i}\lambda_2 x_{w(2)}} \cdots \mathrm{e}^{2\pi\,\mathrm{i}\lambda_n x_{w(n)}} = \sum_{w\in S_n} (\det w)\, \mathrm{e}^{2\pi\,\mathrm{i}\langle\lambda, wx\rangle}, \tag{8}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a set of real numbers. When $\lambda_1, \lambda_2, \ldots, \lambda_n$ are integers, we denote them as $m = (m_1, m_2, \ldots, m_n)$,

$$E_m^-(x) = \det(\mathrm{e}^{2\pi\,\mathrm{i}m_i x_j})_{i,j=1}^n. \tag{9}$$

It is seen from properties of determinants that the functions $E_\lambda^-(x)$ satisfy the relation

$$E_\lambda^-(x_1 + a, x_2 + a, \ldots, x_n + a) = \mathrm{e}^{2\pi\,\mathrm{i}(\lambda_1+\lambda_2+\cdots+\lambda_n)a} E_\lambda^-(x), \tag{10}$$

that is, it is enough to consider functions $E_\lambda^-(x)$ on some hyperplane $\mathcal{H}_b$. As in the case of symmetric exponential functions, a transition from one hyperplane $\mathcal{H}_b$ to another $\mathcal{H}_c$ for the function E_λ^- is fulfilled by means of multiplication by a usual exponential function $\mathrm{e}^{2\pi\,\mathrm{i}|\lambda|(c-b)}$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. For points x of the hyperplane $\mathcal{H}_0$ we have the relation

$$E_{\lambda_1+\nu,\lambda_2+\nu,\ldots,\lambda_n+\nu}^-(x) = E_\lambda^-(x). \tag{11}$$

It follows from properties of determinants that $E_\lambda^-(x) = 0$ if λ has at least two coinciding numbers or if x has at least two coinciding coordinates. For any permutation $w \in S_n$ we receive

$$E_{w\lambda}^-(x) = (\det w) E_\lambda^-(x), \qquad E_\lambda^-(wx) = (\det w) E_\lambda^-(x). \tag{12}$$

This means that it is enough to consider the antisymmetric exponential functions $E_\lambda^-(x)$ for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_n.$$

Such λ are called *strictly dominant*. The set of these λ is denoted by D_+^+.

3. Affine symmetric group and fundamental domains

We have seen that the functions $E_\lambda^+(x)$ are symmetric with respect to the permutation group S_n, that is, $E_\lambda^+(wx) = E_\lambda^+(x)$, $w \in S_n$. The symmetric exponential functions E_m^+ with integral $m = (m_1, m_2, \ldots, m_n)$ admit additional symmetries related to the periodicity of the exponential functions $\mathrm{e}^{2\pi\,iry}$, $r \in \mathbb{Z}$, $y \in \mathbb{R}$. These symmetries are described by the discrete group of shifts in the space E_n by vectors

$$r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + \cdots + r_n\mathbf{e}_n, \qquad r_i \in \mathbb{Z},$$

where $\mathbf{e}, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the unit vectors in directions of the corresponding axes. We denote this group by T_n. Permutations of S_n and shifts of T_n generate a group which is denoted as S_n^{aff} and is called the *affine symmetric group*. The group S_n^{aff} is a semidirect product of its subgroups S_n and T_n,

$$S_n^{\mathrm{aff}} = S_n \times T_n,$$

where T_n is an invariant subgroup, that is, $wtw^{-1} \in T_n$ for $w \in S_n$ and $t \in T_n$.

An open simply connected set $F \subset \mathbb{R}^n$ is called a *fundamental domain* for the group S_n^{aff} (for the group S_n) if it does not contain equivalent points (that is, points x and x', such that $x' = wx$, where w belongs to S_n^{aff} or S_n, respectively) and if its closure contains at least one point from each S_n^{aff}-orbit (from each S_n-orbit). Recall that a S_n^{aff}-orbit of a point $x \in \mathbb{R}^n$ is the set of points wx, $w \in S_n^{\mathrm{aff}}$.

It is evident that the set D_+^+ of all points $x = (x_1, x_2, \ldots, x_n)$, such that

$$x_1 > x_2 > \cdots > x_n$$

constitutes a fundamental domain for the group S_n (we denote it as $F(S_n)$). The set of points $x = (x_1, x_2, \ldots, x_n) \in D_+^+$, such that

$$1 > x_1 > x_2 > \cdots > x_n > 0$$

constitutes a fundamental domain for the affine group S_n^{aff} (we denote it as $F(S_n^{\mathrm{aff}})$).

As we have seen, the functions $E_\lambda^+(x)$ are symmetric with respect to the permutation group S_n. This means that it is enough to consider the functions $E_\lambda^+(x)$ only on the closure of the fundamental domain $F(S_n)$. Values of E_λ^+ on other points are received by using symmetricity.

The symmetricity of functions E_m^+ with integral $m = (m_1, m_2, \ldots, m_n)$ with respect to the affine symmetric group S_n^{aff},

$$E_m^+(wx + r) = E_m^+(x), \qquad w \in S_n, \qquad r \in T_n, \tag{13}$$

means that we may consider $E_m^+(x)$ only on the closure of the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$, that is, on the set of points x, such that $1 \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$. Values of $E_m^+(x)$ on other points are obtained by using relation (13).

The exponential functions $E_m^-(x)$ with integral $m = (m_1, m_2, \ldots, m_n)$ also admit additional symmetries related to the periodicity of the usual exponential functions $\mathrm{e}^{2\pi \mathrm{i} ry}$, $r \in \mathbb{Z}$, $y \in \mathbb{R}$. These symmetries are described by the affine symmetric group S_n^{aff}. We have

$$E_m^-(wx + r) = (\det w) E_m^-(x), \qquad w \in S_n, \qquad r \in T_n, \tag{14}$$

that is, it is enough to consider the functions $E_m^-(x)$ only on the closure of the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$. Values of $E_m^-(x)$ on other points are obtained by using relation (14).

The functions $E_\lambda^-(x)$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{R}$, are antisymmetric with respect to the symmetric group S_n,

$$E_\lambda^-(wx) = (\det w) E_m^-(x), \; w \in S_n.$$

For this reason, we may consider E_λ^- only on the fundamental domain $F(S_n)$.

4. Properties

Symmetricity and antisymmetricity of symmetric and antisymmetric multivariate exponential functions are the main properties of these functions. However, they possess many other interesting properties.

4.1. Behaviour on boundary

The symmetric and antisymmetric functions $E_\lambda^+(x)$ and $E_\lambda^-(x)$ are finite sums of exponential functions. Therefore, they are continuous functions of $x_1, x_2, \ldots, x_n$ and have continuous derivatives of all orders in $\mathbb{R}^n$.

The closure $\overline{F(S_n)}$ of the fundamental domain $F(S_n)$ without points of $F(S_n)$ is called a *boundary* of the fundamental domain $F(S_n)$ and is denoted by $\partial F(S_n)$. A point $x = (x_1, x_2, \ldots, x_n) \in \overline{F(S_n)}$ belongs to $\partial F(S_n)$ if and only if at least two coordinates x_i, x_j

in x coincide. It is clear that the boundary $\partial F(S_n)$ is composed of points of $\overline{F(S_n)}$ belonging to the hyperplanes given by the equations

$$x_i = x_j, \qquad i, j = 1, 2, \ldots, n, \qquad i \neq j.$$

Similarly, the boundary $\partial F\left(S_n^{\mathrm{aff}}\right)$ of the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$ consists of points of $\overline{F\left(S_n^{\mathrm{aff}}\right)}$ which do not belong to $F\left(S_n^{\mathrm{aff}}\right)$. A point $x = (x_1, x_2, \ldots, x_n) \in \overline{F\left(S_n^{\mathrm{aff}}\right)}$ belongs to $\partial F\left(S_n^{\mathrm{aff}}\right)$ if and only if at least two coordinates x_i, x_j in x coincide or if one of the conditions $x_1 = 1, x_n = 0$ is fulfilled.

It follows from the properties of determinants that the function $E_\lambda^-(x)$ vanishes on the boundary $\partial F(S_n)$,

$$E_\lambda^-(x) = 0, \qquad \text{for} \qquad x \in \partial F(S_n).$$

This relation is true for $E_m^-(x), m_i \in \mathbb{Z}$. In this case, we also have $E_m^-(x) = 0$ for points $x \in \partial F\left(S_n^{\mathrm{aff}}\right)$, such that $x_1 - x_n = 1$.

For the symmetric multivariate functions $E_\lambda^+(x)$ we have

$$\frac{\partial E_\lambda^+(x)}{\partial \mathbf{n}} = 0 \qquad \text{for} \qquad x \in \partial F(S_n),$$

where $\mathbf{n}$ is the normal to the boundary $\partial F(S_n)$.

4.2. Complex conjugation

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be a strictly dominant element, that is, $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. We have

$$E_\lambda^-(x) = \sum_{w \in S_n} (\det w)\, \mathrm{e}^{2\pi\, \mathrm{i}((w\lambda)_1 x_1 + \cdots + (w\lambda)_n x_n)}, \tag{15}$$

where $(w\lambda)_1, (w\lambda)_2, \ldots, (w\lambda)_n$ are the coordinates of the point $w\lambda$.

The element $-(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ is strictly dominant if the element $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is strictly dominant. In the group S_n there exists an element w_0, such that

$$w_0(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_n, \lambda_{n-1}, \ldots, \lambda_1).$$

It is easy to calculate that the set $(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$ is obtained from $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ by $n(n-1)/2$ permutations of two neighbouring numbers. Clearly, $\det w_0 = 1$ if this number is even and $\det w_0 = -1$ otherwise. Thus,

$$\det w_0 = 1 \qquad \text{for} \qquad n = 4k \qquad \text{and} \qquad n = 4k+1,$$
$$\det w_0 = -1 \qquad \text{for} \qquad n = 4k-2 \qquad \text{and} \qquad n = 4k-1,$$

where k is an integer. It follows from here that in the expressions for the exponential functions $E^-_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x)$ and $E^-_{-(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(x)$ there are summands

$$\mathrm{e}^{2\pi\, \mathrm{i}\langle w_0\lambda, x\rangle} = \mathrm{e}^{2\pi\, \mathrm{i}(\lambda_n x_1 + \cdots + \lambda_1 x_n)} \qquad \text{and} \qquad \mathrm{e}^{-2\pi\, \mathrm{i}(\lambda_n x_1 + \cdots + \lambda_1 x_n)}, \tag{16}$$

respectively, which are complex conjugate to each other. Moreover, the first expression is contained with the sign $(\det w_0)$ in $E^-_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x)$, that is, expressions (16) are contained in $E^-_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x)$ and $E^-_{-(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(x)$ with the same sign for $n = 4k, 4k+1$ and with opposite signs for $n = 4k-2, 4k-1, k \in \mathbb{Z}$.

Similarly, in expressions (15) for the function $E^-_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x)$ and for the function $E^-_{-(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(x)$ all other summands are (up to a sign, which depends on a value of n) pairwise complex conjugate. Therefore,

$$E^-_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x) = \overline{E^-_{-(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(x)} \tag{17}$$

for $n = 4k, 4k + 1$ and

$$E^{-}_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x) = -\overline{E^{-}_{-(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(x)} \tag{18}$$

for $n = 4k - 2, 4k - 1$.

According to (17) and (18), if

$$(\lambda_1, \lambda_2, \ldots, \lambda_n) = -(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1), \tag{19}$$

then *the function E^{-}_{λ} is real for $n = 4k, 4k + 1$ and pure imaginary for $n = 4k - 2, 4k - 1$.* Moreover, the right-hand side of (15) for this case consists of pairs of terms which give sine or cosine functions. It is representable as a sum of cosines of angles if $n = 4k, 4k + 1$ and as a sum of sines of angles multiplied by $\mathrm{i} = \sqrt{-1}$ if $n = 4k - 2, 4k - 1$.

It is proved similarly that for the symmetric exponential functions $E^{+}_{\lambda}(x)$ and $E^{+}_{w_0\lambda}(x)$ we have the following relation:

$$E^{+}_{(\lambda_1,\lambda_2,\ldots,\lambda_n)}(x) = \overline{E^{+}_{-(\lambda_n,\lambda_{n-1},\ldots,\lambda_1)}(x)}.$$

If $(\lambda_1, \lambda_2, \ldots, \lambda_n) = -(\lambda_n, \lambda_{n-1}, \ldots, \lambda_1)$, then the function $E^{+}_{\lambda}(x)$ is real. In this case, $E^{+}_{\lambda}(x)$ can be represented as a sum of cosines of the corresponding angles.

4.3. Scaling symmetry

For $c \in \mathbb{R}$, let $c\lambda = (c\lambda_1, c\lambda_2, \ldots, c\lambda_n)$. Then

$$E^{-}_{c\lambda}(x) = \sum_{w\in W} (\det w)\, \mathrm{e}^{2\pi\,\mathrm{i}\langle cw\lambda, x\rangle} = \sum_{w\in W} (\det w)\, \mathrm{e}^{2\pi\,\mathrm{i}\langle w\lambda, cx\rangle} = E^{-}_{\lambda}(cx).$$

The equality $E^{-}_{c\lambda}(x) = E^{-}_{\lambda}(cx)$ expresses the *scaling symmetry of exponential functions* $E^{-}_{c\lambda}(x)$. The scaling symmetry is true also for symmetric exponential functions, $E^{+}_{c\lambda}(x) = E^{+}_{\lambda}(cx)$.

4.4. Duality

Due to invariance of the scalar product $\langle\cdot,\cdot\rangle$ with respect to the symmetric group S_n, $\langle w\mu, wy\rangle = \langle\mu, y\rangle$, for $x = (x_1, x_2, \ldots, x_n)$, $x_i \neq x_j$, $i \neq j$, we have

$$E^{-}_{\lambda}(x) = \sum_{w\in W} (\det w)\, \mathrm{e}^{2\pi\,\mathrm{i}\langle\lambda, w^{-1}x\rangle} = \sum_{w\in W} (\det w)\, \mathrm{e}^{2\pi\,\mathrm{i}\langle x, w\lambda\rangle} = E^{-}_{x}(\lambda).$$

This relation expresses the *duality* of antisymmetric orbit functions. The duality is true also for the symmetric exponential functions, $E^{+}_{\lambda}(x) = E^{+}_{x}(\lambda)$.

4.5. Orthogonality on the fundamental domain $F\left(S^{\mathrm{aff}}_n\right)$

The antisymmetric exponential functions E^{-}_{m} with $m = (m_1, m_2, \ldots, m_n) \in D^{+}_{+}$, $m_j \in \mathbb{Z}$, are orthogonal on $F\left(S^{\mathrm{aff}}_n\right)$ with respect to the Euclidean measure,

$$\left|F\left(S^{\mathrm{aff}}_n\right)\right|^{-1} \int_{F(S^{\mathrm{aff}}_n)} E^{-}_{m}(x)\overline{E^{-}_{m'}(x)}\, \mathrm{d}x = |S_n|\delta_{mm'}, \tag{20}$$

where the overbar means complex conjugation, $|S_n|$ means a number of elements in the set S_n and $\left|F\left(S^{\mathrm{aff}}_n\right)\right|$ is an area of the fundamental domain $F\left(S^{\mathrm{aff}}_n\right)$. This relation follows from the equality

$$\int_{\mathsf{T}} E^{-}_{m}(x)\overline{E^{-}_{m'}(x)}\, \mathrm{d}x = |S_n|\delta_{mm'}$$

(where T is the torus in E_n consisting of points $x = (x_1, x_2, \ldots, x_n)$, $0 \leqslant x_i < 1$), which is a consequence of orthogonality of the exponential functions $\mathrm{e}^{2\pi\,\mathrm{i}\langle\mu,x\rangle}$ (entering into the definition of $E^-_m(x)$) for different sets μ.

If to assume that an area of T is equal to 1, $|\mathsf{T}| = 1$, then $\left|F\left(S_n^{\text{aff}}\right)\right| = |S_n|^{-1}$ and formula (20) takes the form

$$\int_{F(S_n^{\text{aff}})} E^-_m(x)\overline{E^-_{m'}(x)}\,\mathrm{d}x = \delta_{mm'}. \tag{21}$$

In expression (3) for symmetric exponential functions there can be coinciding summands. For this reason, for the symmetric exponential functions E^+_m with $m = (m_1, m_2, \ldots, m_n) \in D_+$, $m_j \in \mathbb{Z}$, relation (21) is replaced by

$$\int_{F(S_n^{\text{aff}})} E^+_m(x)\overline{E^+_{m'}(x)}\,\mathrm{d}x = |S_m|\delta_{mm'}, \tag{22}$$

where $|S_m|$ is a number of elements in the subgroup S_m of S_n consisting of elements $w \in S_n$, such that $wm = m$.

4.6. Orthogonality of symmetric and antisymmetric exponential functions.

Let w_i $(i = 1, 2, \ldots, n-1)$ be the permutation of coordinates x_i and x_{i+1}. We create the domain $F^{\text{ext}}\left(S_n^{\text{aff}}\right) = F\left(S_n^{\text{aff}}\right) \cup w_i F\left(S_n^{\text{aff}}\right)$, where $F\left(S_n^{\text{aff}}\right)$ is the fundamental domain for the affine group S_n^{aff}. Since for $m = (m_1, m_2, \ldots, m_n) \in D^+_+$, $m_j \in \mathbb{Z}$, we have $E^+_m(w_i x) = E^+_m(x)$ and $E^-_m(w_i x) = -E^-_m(x)$, then

$$\int_{F^{\text{ext}}(S_n^{\text{aff}})} E^+_m(x)\overline{E^-_{m'}(x)}\,\mathrm{d}x = 0. \tag{23}$$

Indeed, due to symmetry and antisymmetry of symmetric and antisymmetric exponential functions, respectively, we have

$$\begin{aligned}\int_{F^{\text{ext}}(S_n^{\text{aff}})} E^+_m(x)\overline{E^-_{m'}(x)}\,\mathrm{d}x &= \int_{F(S_n^{\text{aff}})} E^+_m(x)\overline{E^-_{m'}(x)}\,\mathrm{d}x + \int_{w_i F(S_n^{\text{aff}})} E^+_m(x)\overline{E^-_{m'}(x)}\,\mathrm{d}x \\ &= \int_{F(S_n^{\text{aff}})} E^+_m(x)\overline{E^-_{m'}(x)}\,\mathrm{d}x + \int_{F(S_n^{\text{aff}})} E^+_m(x)\overline{(-E^-_{m'}(x))}\,\mathrm{d}x = 0.\end{aligned}$$

Relation (23) is a generalization of the orthogonality of the functions sine and cosine on the interval $(0, 2\pi)$.

5. Special cases

The special case of symmetric and antisymmetric exponential functions at $\lambda = \frac{1}{2}(n-1, n-3, \ldots, -n+3, -n+1) \equiv \rho$ is of great interest since it is met in the representation theory. The antisymmetric exponential function $E^-_\rho(x)$ is given by the formula

$$E^-_\rho(x) = (2\mathrm{i})^{n(n-1)/2} \prod_{1\leqslant i<j\leqslant n} \sin\pi(x_i - x_j). \tag{24}$$

It follows if to represent $\sin\pi(x_i - x_j)$ in terms of exponential functions, then to fulfil multiplication of these functions and to compare with expression (8) for $E^-_\rho(x)$.

Let us set $(m_1, m_2, \ldots, m_n) = (n-1, n-2, \ldots, 1, 0) \equiv \rho'$. The antisymmetric exponential function $E^-_{\rho'}(x)$ can be written in the form of the Vandermonde determinant,

$$E^-_{\rho'}(x) = \det\left(\mathrm{e}^{2\pi\,\mathrm{i}(n-i)x_j}\right)_{i,j=1}^n = \prod_{k<l}\left(\mathrm{e}^{2\pi\,\mathrm{i}x_k} - \mathrm{e}^{-2\pi\,\mathrm{i}x_l}\right). \tag{25}$$

The last equality follows from the expression for the Vandermonde determinant. Since $\rho' = \rho + \frac{n-1}{2}$, expressions (24) and (25) are connected by the relation

$$E^-_{\rho'}(x) = \mathrm{e}^{\pi\,\mathrm{i}|x|(n-1)} E^-_{\rho}(x),$$

where $|x| = \sum_{i=1}^n x_i$.

It is easy to see that *the function* $E^-_\rho(x)$ does not vanish on intrinsic points of the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$.

The symmetric counterpart $E^+_\rho(x)$ of formula (24) for the antisymmetric exponential function $E^-_\rho(x)$ has the form

$$E^+_\rho(x) = 2^{n(n-1)/2} \prod_{1\leqslant i<j\leqslant n} \cos\pi(x_i - x_j). \tag{26}$$

6. Solutions of the Laplace equation

The Laplace operator on the n-dimensional Euclidean space E_n in the orthogonal coordinates $x = (x_1, x_2, \ldots, x_n)$ has the form

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We take any summand in the expression for symmetric or antisymmetric multivariate exponential function and act upon it by the operator Δ. We get

$$\Delta\, \mathrm{e}^{2\pi\,\mathrm{i}((w(\lambda))_1 x_1+\cdots+(w(\lambda))_n x_n)} = -4\pi^2\langle\lambda,\lambda\rangle\, \mathrm{e}^{2\pi\,\mathrm{i}((w(\lambda))_1 x_1+\cdots+(w(\lambda))_n x_n)},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ determines $E^-_\lambda(x)$. Since this action of Δ does not depend on a summand from the expression for symmetric or antisymmetric exponential function, we have

$$\Delta E^-_\lambda(x) = -4\pi^2\langle\lambda,\lambda\rangle E^-_\lambda(x), \qquad \Delta E^+_\lambda(x) = -4\pi^2\langle\lambda,\lambda\rangle E^+_\lambda(x). \tag{27}$$

Formula (27) can be generalized in the following way. Let $\sigma_k(y_1, y_2, \ldots, y_n)$ be the kth elementary symmetric polynomial of degree k, that is,

$$\sigma_k(y_1, y_2, \ldots, y_n) = \sum_{1\leqslant k_1<k_2<\cdots<k_n\leqslant n} y_{k_1} y_{k_2}\cdots y_{k_n}.$$

Then for $k = 1, 2, \ldots, n$ we have

$$\sigma_k\left(\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \ldots, \frac{\partial^2}{\partial x_n^2}\right) E^\pm_\lambda(x) = (-4\pi^2)^k \sigma_k\left(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2\right) E^\pm_\lambda(x). \tag{28}$$

Note that n differential equations (28) are algebraically independent.

Thus, the antisymmetric exponential functions $E^-_\lambda(x)$ are eigenfunctions of the operators $\sigma_k\left(\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \ldots, \frac{\partial^2}{\partial x_n^2}\right)$, $k = 1, 2, \ldots, n$, on the fundamental domain $F(S_n)$ of the symmetric group S_n satisfying the boundary condition

$$E^-_m(x) = 0 \qquad \text{for} \qquad x \in \partial F(S_n). \tag{29}$$

Similarly, the symmetric exponential functions $E^+_\lambda(x)$ are eigenfunctions of the operators $\sigma_k\left(\frac{\partial^2}{\partial x_1^2}, \frac{\partial^2}{\partial x_2^2}, \ldots, \frac{\partial^2}{\partial x_n^2}\right)$, $k = 1, 2, \ldots, n$, on the fundamental domain $F(S_n)$ satisfying the boundary condition

$$\frac{\partial E^+_m(x)}{\partial \mathbf{n}} = 0 \qquad \text{for} \quad x \in \partial F(S_n),$$

where $\mathbf{n}$ is the normal to the boundary $\partial F(S_n)$. That is, they are the solutions of the Neumann boundary value problem for the domain $F(S_n)$.

7. Expansions in (anti)symmetric exponential functions on $F\big(S_n^{\mathrm{aff}}\big)$

Symmetric and antisymmetric exponential functions determine symmetric and antisymmetric multivariate Fourier transforms which generalize the usual Fourier transform.

As in the case of exponential functions of one variable, (anti)symmetric exponential functions determine three types of Fourier transforms:

(a) Fourier transforms related to the functions $E_m^{\pm}(x)$ with $m = (m_1, m_2, \ldots, m_n), m_j \in \mathbb{Z}$ (Fourier series);
(b) Fourier transforms related to $E_\lambda^{\pm}(x)$ with $\lambda \in D_+$;
(c) symmetrized and antisymmetrized multivariate finite Fourier transforms.

In this section, we consider expansions in (anti)symmetric exponential functions on the fundamental domain $F\big(S_n^{\mathrm{aff}}\big)$.

Let $f(x)$ be a symmetric (with respect to the affine symmetric group S_n^{aff}) continuous function on the n-dimensional Euclidean space E_n which has continuous derivatives. We may consider this function on the torus T which is a closure of the union of the sets $wF\big(S_n^{\mathrm{aff}}\big)$, $w \in S_n$. The function $f(x)$, as a function on T, can be expanded in the exponential functions $\mathrm{e}^{2\pi\,\mathrm{i}m_1x_1}\,\mathrm{e}^{2\pi\,\mathrm{i}m_2x_2}\cdots\mathrm{e}^{2\pi\,\mathrm{i}m_nx_n}$, $m_i \in \mathbb{Z}$. We have

$$f(x) = \sum_{m_i\in\mathbb{Z}} c_m\,\mathrm{e}^{2\pi\,\mathrm{i}m_1x_1}\,\mathrm{e}^{2\pi\,\mathrm{i}m_2x_2}\cdots\mathrm{e}^{2\pi\,\mathrm{i}m_nx_n}, \tag{30}$$

where $m = (m_1, m_2, \ldots, m_n)$. Due to the symmetry $f(wx) = f(x)$, $w \in S_n$, for any $w \in S_n$ we have

$$\begin{aligned} f(wx) &= \sum_{m_i\in\mathbb{Z}} c_m\,\mathrm{e}^{2\pi\,\mathrm{i}m_1x_{w(1)}}\cdots\mathrm{e}^{2\pi\,\mathrm{i}m_nx_{w(n)}} = \sum_{m_i\in\mathbb{Z}} c_m\,\mathrm{e}^{2\pi\,\mathrm{i}m_{w^{-1}(1)}x_1}\cdots\mathrm{e}^{2\pi\,\mathrm{i}m_{w^{-1}(n)}x_n} \\ &= \sum_{m_i\in\mathbb{Z}} c_{wm}\,\mathrm{e}^{2\pi\,\mathrm{i}m_1x_1}\cdots\mathrm{e}^{2\pi\,\mathrm{i}m_nx_n} = f(x) = \sum_{m_i\in\mathbb{Z}} c_m\,\mathrm{e}^{2\pi\,\mathrm{i}m_1x_1}\cdots\mathrm{e}^{2\pi\,\mathrm{i}m_nx_n}. \end{aligned}$$

Therefore, the coefficients c_m satisfy the conditions $c_{wm} = c_m$, $w \in S_n$. Collecting in (30) exponential functions at the same c_{wm}, $w \in S_n$, we obtain the expansion

$$f(x) = \sum_{m\in P_+} c_m E_m^+(x), \tag{31}$$

where $P_+ = D_+ \cap \mathbb{Z}^n$. Thus, *any symmetric (with respect to S_n) continuous function f on T which has continuous derivatives (that is, any continuous function on $F\big(S_n^{\mathrm{aff}}\big)$ with continuous derivatives) can be expanded in the symmetric exponential functions $E_m^+(x)$, $m \in P_+$.*

By the orthogonality relation (22), the coefficients c_m in expansion (31) are determined by the formula

$$c_m = |S_m|^{-1}\int_{F(S_n^{\mathrm{aff}})} f(x)\overline{E_m^+(x)}\,\mathrm{d}x, \tag{32}$$

where, as before, $|S_m|$ is a number of elements in the subgroup S_m of S_n consisting of $w \in S_n$, such that $wm = m$. Moreover, the Plancherel formula

$$\sum_{m\in P^+} |c_m|^2 = |S_m|^{-1}\int_{F(S_n^{\mathrm{aff}})} |f(x)|^2\mathrm{d}x \tag{33}$$

holds, which means that the Hilbert spaces with the appropriate scalar products are isometric.

Formula (32) is the symmetrized Fourier transform of the function $f(x)$. Formula (31) gives an inverse transform. Formulae (31) and (32) give the *symmetric multivariate Fourier transforms* corresponding to the symmetric exponential functions $E_m^+(x)$, $m \in P^+$.

Analogous transforms hold for the antisymmetric exponential functions $E^-_m(x), m \in P^+_+ \equiv D^+_+ \cap \mathbb{Z}^n$. Let $f(x)$ be an antisymmetric (with respect to the symmetric group S_n) continuous function on the n-dimensional torus T, which has continuous derivatives. We may consider this function as a function on $F\left(S_n^{\text{aff}}\right)$. Then we have the expansion

$$f(x) = \sum_{m\in P^+_+} c_m E^-_m(x), \qquad \text{where} \quad c_m = \int_{F(S_n^{\text{aff}})} f(x)\overline{E^-_m(x)}\,\mathrm{d}x. \tag{34}$$

Moreover, the Plancherel formula holds

$$\sum_{m\in P^+_+} |c_m|^2 = \int_{F(S_n^{\text{aff}})} |f(x)|^2 \mathrm{d}x. \tag{35}$$

Let $\mathcal{L}^2_0\left(F\left(S_n^{\text{aff}}\right)\right)$ denote the Hilbert space of functions on the fundamental domain $F\left(S_n^{\text{aff}}\right)$, which behaves on the boundary $\partial F\left(S_n^{\text{aff}}\right)$ of the fundamental domain $F\left(S_n^{\text{aff}}\right)$ in the same way as the functions $E^-_m(x)$ do. Let

$$\langle f_1, f_2\rangle = \int_{F(S_n^{\text{aff}})} f_1(x)\overline{f_2(x)}\,\mathrm{d}x$$

be a scalar product in this space. Formulae (34) and (35) show that *the set of exponential functions* $E^-_m(x), m \in P^+_+$, *forms an orthogonal basis of* $\mathcal{L}^2_0\left(F\left(S_n^{\text{aff}}\right)\right)$.

Let $F^{\text{ext}}\left(S_n^{\text{aff}}\right) = F\left(S_n^{\text{aff}}\right) \bigcup F\left(w_i S_n^{\text{aff}}\right)$ denote the set from section 4. Then we can extend the symmetric and antisymmetric Fourier transforms to the functions from the Hilbert space $\mathcal{L}^2\left(F^{\text{ext}}\left(S_n^{\text{aff}}\right)\right)$ with the scalar product

$$\langle f_1, f_2\rangle = \int_{F^{\text{ext}}(S_n^{\text{aff}})} f_1(x)\overline{f_2(x)}\,\mathrm{d}x.$$

This transform is of the form

$$f(x) = \sum_{m\in P_+} c_m E^+_m(x) + \sum_{m\in P^+_+} c'_m E^-_m(x), \tag{36}$$

where

$$c_m = |S_m|^{-1} \int_{F(S_n^{\text{aff}})} f(x)\overline{E^+_m(x)}\,\mathrm{d}x, \qquad c'_m = \int_{F(S_n^{\text{aff}})} f(x)\overline{E^-_m(x)}\,\mathrm{d}x. \tag{37}$$

The corresponding Plancherel formula holds. The functions $E^+_m(x), m \in P_+$ and $E^-_m(x), m \in P^+_+$, form a complete orthogonal basis of the Hilbert space $\mathcal{L}^2\left(F^{\text{ext}}\left(S_n^{\text{aff}}\right)\right)$.

8. Multivariate Fourier transforms on the fundamental domain $F(S_n)$

Expansions (31) and (34) of functions on the fundamental domain $F\left(S_n^{\text{aff}}\right)$ are respectively expansions in the symmetric and antisymmetric exponential functions $E^+_m(x)$ and $E^-_m(x)$ with integral $m = (m_1, m_2, \ldots, m_n)$. The exponential functions $E^+_\lambda(x)$ and $E^-_\lambda(x)$ with λ lying in the fundamental domain $F(S_n)$ (and not obligatory integral) are not invariant (anti-invariant) with respect to the corresponding affine symmetric group S_n^{aff}. They are invariant (anti-invariant) only with respect to the permutation group S_n. A fundamental domain of S_n coincides with the set D^+_+ consisting of the points x, such that $x_1 > x_2 > \cdots > x_n$. For this reason, the functions $E^-_\lambda(x), \lambda \in D^+_+$, and $E^+_\lambda(x), \lambda \in D_+$, determine Fourier transforms on D_+.

We began with the usual Fourier transforms on $\mathbb{R}^n$:

$$\tilde{f}(\lambda) = \int_{\mathbb{R}^n} f(x)\,\mathrm{e}^{2\pi\,\mathrm{i}\langle\lambda,x\rangle}\,\mathrm{d}x, \tag{38}$$

$$f(x) = \int_{\mathbb{R}^n} \tilde{f}(\lambda)\, \mathrm{e}^{-2\pi\, \mathrm{i}\langle\lambda, x\rangle}\, \mathrm{d}\lambda. \tag{39}$$

Let the function $f(x)$ be anti-invariant with respect to the symmetric group S_n, that is, $f(wx) = (\det w) f(x)$, $w \in S_n$. It is easy to check that $\tilde{f}(\lambda)$ is also anti-invariant with respect to the group S_n. Replace λ by $w\lambda$ in (38), $w \in S_n$, multiply both sides by $\det w$, and sum these both sides over $w \in S_n$. Then instead of (38) we obtain

$$\tilde{f}(\lambda) = \int_{D_+} f(x) E_\lambda^-(x)\, \mathrm{d}x, \qquad \lambda \in D_+^+, \tag{40}$$

where we have taken into account that $f(x)$ is anti-invariant with respect to S_n.

Similarly, starting from (39), we obtain the inverse formula

$$f(x) = \int_{D_+} \tilde{f}(\lambda) \overline{E_\lambda^-(x)}\, \mathrm{d}\lambda. \tag{41}$$

For the transforms (40) and (41) the Plancherel formula

$$\int_{D_+} |f(x)|^2 \mathrm{d}x = \int_{D_+} |\tilde{f}(\lambda)|^2 \mathrm{d}\lambda$$

holds. Formulae (40) and (41) determine the *antisymmetric multivariate Fourier transform on the domain* $F(S_n)$.

Similarly, starting from formulae (38) and (39) we receive the symmetric multivariate Fourier transform on the domain $F(S_n)$:

$$\tilde{f}(\lambda) = \int_{D_+} f(x) E_\lambda^+(x)\, \mathrm{d}x, \qquad f(x) = \int_{D_+} \tilde{f}(\lambda) \overline{E_\lambda^+(x)}\, \mathrm{d}\lambda. \tag{42}$$

The corresponding Plancherel formula holds.

9. Finite Fourier transform

Along with the integral Fourier transform in one variable there exists a discrete Fourier transform in one variable running over a finite set. Similarly, it is possible to introduce finite multivariate antisymmetric and symmetric Fourier transforms, based on antisymmetric and symmetric exponential functions. We first consider the finite Fourier transform in one variable, which will be used below. In the following sections, we expose a general antisymmetric and symmetric Fourier transforms (under exposition we shall use the methods developed in [16]).

Let us fix a positive integer N and consider the numbers

$$e_{mn} := N^{-1/2} \exp(2\pi\, \mathrm{i} mn/N), \qquad m, n = 1, 2, \ldots, N. \tag{43}$$

The matrix $(e_{mn})_{m,n=1}^N$ is unitary, that is,

$$\sum_k e_{mk} \overline{e_{nk}} = \delta_{mn}, \qquad \sum_k e_{km} \overline{e_{kn}} = \delta_{mn}. \tag{44}$$

Indeed, according to the formula for a sum of a geometric progression we have

$$t^a + t^{a+1} + \cdots + t^{a+r} = (1-t)^{-1} t^a (1 - t^{r+1}), \qquad t \neq 1,$$
$$t^a + t^{a+1} + \cdots + t^{a+r} = r + 1, \qquad t = 1.$$

Setting $t = \exp(2\pi\, \mathrm{i}(m-n)/N)$, $a = 1$ and $r = N - 1$, we prove (44).

Let $f(n)$ be a function of $n \in \{1, 2 \ldots, N\}$. We may consider the transform

$$\sum_{n=1}^N f(n) e_{mn} \equiv N^{-1/2} \sum_{n=1}^N f(n) \exp(2\pi \mathrm{i} mn/N) = \tilde{f}(m). \tag{45}$$

Then due to unitarity of the matrix $(e_{mn})_{m,n=1}^{N}$, we express $f(n)$ as a linear combination of conjugates of the functions (43):

$$f(n) = N^{-1/2}\sum_{m=1}^{N}\tilde f(m)\exp(-2\pi\,\mathrm{i}mn/N). \tag{46}$$

The function $\tilde f(m)$ is a *finite Fourier transform* of $f(n)$. This transform is a linear map. Formula (46) gives an inverse transform. The Plancherel formula

$$\sum_{m=1}^{N}|\tilde f(m)|^2 = \sum_{n=1}^{N}|f(n)|^2$$

holds for transforms (45) and (46). This means that the finite Fourier transform conserves the norm introduced in the space of functions on $\{1, 2, \ldots, N\}$.

10. Antisymmetric multivariate discrete Fourier transforms

We use the discrete exponential function (43),

$$e_m(s) := N^{-1/2}\exp(2\pi\,\mathrm{i}ms), \qquad s \in F_N \equiv \left\{\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\right\}, \qquad m \in \mathbb{Z}^{\geqslant 0}, \tag{47}$$

and make a multivariate discrete exponential function by taking a product of n copies of these functions,

$$\begin{aligned} e_{\mathbf{m}}(\mathbf{s}) &:= e_{m_1}(s_1)e_{m_2}(s_2)\cdots e_{m_n}(s_n) \\ &= N^{-n/2}\exp(2\pi\,\mathrm{i}m_1s_1)\exp(2\pi\,\mathrm{i}m_2s_2)\cdots\exp(2\pi\,\mathrm{i}m_ns_n), \end{aligned} \tag{48}$$

where $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in F_N^n$ and $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in (\mathbb{Z}^{\geqslant 0})^n$. Now we take these multivariate functions for integers m_i, such that $m_1 > m_2 > \cdots > m_n \geqslant 0$ and make an antisymmetrization. As a result, we obtain a finite version of the antisymmetric exponential functions (8),

$$\tilde E^-_{\mathbf{m}}(\mathbf{s}) := |S_n|^{-1/2}\det(e_{m_i}(s_j))_{i,j=1}^{n} = |S_n|^{-1/2}N^{-n/2}E^-_{\mathbf{m}}(\mathbf{s}), \tag{49}$$

where, as before, $|S_n|$ is the order of the symmetric group S_n.

The n-tuples $\mathbf{s}$ in (49) run over $F_N^n \equiv F_N \times \cdots \times F_N$ (n times). We denote by $\hat F_N^n$ the subset of F_N^n consisting of $\mathbf{s} \in F_N^n$, such that

$$s_1 > s_2 > \cdots > s_n.$$

The set $\hat F_N^n$ is a finite subset of the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$ of the group S_n^{aff}.

Note that acting by permutations $w \in S_n$ upon $\hat F_N^n$ we obtain the whole set F_N^n without those points which are invariant under some nontrivial permutation $w \in S_n$. Clearly, the function (49) vanishes on the last points.

Since the discrete exponential functions $e_m(s)$ satisfy the equality $e_m(s) = e_{m+N}(s)$, we do not need to consider them for all values $m \in \mathbb{Z}^{\geqslant 0}$. It is enough to consider them for $m \in \{1, 2, \ldots, N\}$. By $\hat D_N^n$ we denote the set of integer n-tuples $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, such that

$$N \geqslant m_1 > m_2 > \cdots > m_n > 0.$$

We need a scalar product in the space of linear combinations of the functions (48). It is natural to give it by the formula

$$\langle e_{\mathbf{m}}(\mathbf{s}), e_{\mathbf{m}'}(\mathbf{s})\rangle \equiv \prod_{i=1}^{n}\langle e_{m_i}(s_i), e_{m_i'}(s_i)\rangle := \prod_{i=1}^{n}\sum_{s_i\in F_N} e_{m_i}(s_i)\overline{e_{m_i'}(s_i)} = \delta_{\mathbf{m}\mathbf{m}'}, \tag{50}$$

where $m_i, m_i' \in \{1, 2, \ldots, N\}$. Here we used relation (44).

Proposition 1. *For* $\mathbf{m}, \mathbf{m}' \in \hat{D}_N^n$ *the discrete functions (49) satisfy the orthogonality relation*

$$\langle \tilde{E}_{\mathbf{m}}^-(\mathbf{s}), \tilde{E}_{\mathbf{m}'}^-(\mathbf{s})\rangle = |S_n| \sum_{\mathbf{s}\in\hat{F}_N^n} \tilde{E}_{\mathbf{m}}^-(\mathbf{s})\overline{\tilde{E}_{\mathbf{m}'}^-(\mathbf{s})} = \delta_{\mathbf{m}\mathbf{m}'}, \tag{51}$$

where the scalar product is determined by formula (50).

Proof. Since $m_1 > m_2 > \cdots > m_n > 0$ and $m_1' > m_2' > \cdots > m_n' > 0$, then due to the definition of the scalar product we have

$$\begin{aligned}\langle \tilde{E}_{\mathbf{m}}^-(\mathbf{s}), \tilde{E}_{\mathbf{m}'}^-(\mathbf{s})\rangle &= \sum_{\mathbf{s}\in F_N^n} \tilde{E}_{\mathbf{m}}^-(\mathbf{s})\overline{\tilde{E}_{\mathbf{m}'}^-(\mathbf{s})} \\ &= |S_n|^{-1} \sum_{w\in S_n}\prod_{i=1}^{n}\sum_{s_i\in F_N} e_{m_{w(i)}}(s_i)\overline{e_{m'_{w(i)}}(s_i)} = \delta_{\mathbf{m}\mathbf{m}'},\end{aligned} \tag{52}$$

where $(m_{w(1)}, m_{w(2)}, \ldots, m_{w(n)})$ is obtained from $(m_1, m_2, \ldots, m_n)$ by action by the permutation $w \in S_n$. Since functions $\tilde{E}_{\mathbf{m}}^-(\mathbf{s})$ are antisymmetric with respect to S_n, then

$$\sum_{\mathbf{s}\in F_N^n} \tilde{E}_{\mathbf{m}}^-(\mathbf{s})\overline{\tilde{E}_{\mathbf{m}'}^-(\mathbf{s})} = |S_n| \sum_{\mathbf{s}\in\hat{F}_N^n} \tilde{E}_{\mathbf{m}}^-(\mathbf{s})\overline{\tilde{E}_{\mathbf{m}'}^-(\mathbf{s})}.$$

This proves the proposition. □

Let f be a function on $\hat{F}_N^n$ (or an antisymmetric function on F_N^n). Then it can be expanded in the functions (49) as

$$f(\mathbf{s}) = \sum_{\mathbf{m}\in\hat{D}_N^n} a_{\mathbf{m}}\tilde{E}_{\mathbf{m}}^-(\mathbf{s}). \tag{53}$$

The coefficients $a_{\mathbf{m}}$ are determined by the formula

$$a_{\mathbf{m}} = |S_n| \sum_{\mathbf{m}\in\hat{F}_N^n} f(\mathbf{s})\overline{\tilde{E}_{\mathbf{m}}^-(\mathbf{s})}. \tag{54}$$

We have taken into account the facts that numbers of elements in $\hat{D}_N^n$ and $\hat{F}_N^n$ are the same and that the discrete functions (49) are orthogonal with respect to the scalar product (51). We call expansions (53) and (54) the *antisymmetric multivariate discrete Fourier transforms*. These expansions can be written in terms of the exponential function $E_{\mathbf{m}}^-(\mathbf{s}) = \det(\exp(2\pi\, \mathrm{i} m_i s_j))_{i,j=1}^n$,

$$f(\mathbf{s}) = N^{-n/2} \sum_{\mathbf{m}\in\hat{D}_N^n} a_{\mathbf{m}} E_{\mathbf{m}}^-(\mathbf{s}), \qquad a_{\mathbf{m}} = N^{-n/2}|S_n| \sum_{\mathbf{m}\in\hat{F}_N^n} f(\mathbf{s})\overline{E_{\mathbf{m}}^-(\mathbf{s})}. \tag{55}$$

11. Symmetric multivariate discrete Fourier transforms

Let us give a symmetric multivariate discrete Fourier transform. For this we take the multivariate exponential functions (48) for integers m_i, such that

$$N \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 1$$

and make a symmetrization. We obtain a finite version of the symmetric exponential functions (3),

$$\tilde{E}_{\mathbf{m}}^+(\mathbf{s}) := |S_n|^{-1/2}\det{}^+(e_{m_i}(s_j))_{i,j=1}^n = |S_n|^{-1/2}N^{-n/2}E_{\mathbf{m}}^+(\mathbf{s}), \tag{56}$$

where the discrete functions $e_m(s)$ are given by (47).

The n-tuples $\mathbf{s}$ in (56) run over $F_N^n \equiv F_N \times \cdots \times F_N$ (n times). We denote by $\check{F}_N^n$ the subset of F_N^n consisting of $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in F_N^n$, such that

$$s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n.$$

The set $\check{F}_N^n$ is a finite subset of the closure of the fundamental domain $F\left(S_n^{\mathrm{aff}}\right)$.

Note that acting by permutations $w \in S_n$ upon $\check{F}_N^n$ we obtain the whole set F_N^n, where each point, having some coordinates m_i coinciding, is repeated several times. Namely, a point $\mathbf{s}$ is contained $|S_{\mathbf{s}}|$ times in $\left\{w\check{F}_N^n;\, w \in S_n\right\}$, where $S_{\mathbf{s}}$ is the subgroup of S_n consisting of elements $w \in S_n$, such that $w\mathbf{s} = \mathbf{s}$.

By $\check{D}_N^n$ we denote the set of integer n-tuples $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, such that

$$N \geqslant m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 1.$$

Proposition 2. *For* $\mathbf{m}, \mathbf{m}' \in \check{D}_N^n$ *the discrete functions (56) satisfy the orthogonality relation*

$$\left\langle \tilde{E}^+_{\mathbf{m}}(\mathbf{s}), \tilde{E}^+_{\mathbf{m}'}(\mathbf{s})\right\rangle = |S_n| \sum_{\mathbf{s}\in\check{F}_M^n} |S_{\mathbf{s}}|^{-1} \tilde{E}^+_{\mathbf{m}}(\mathbf{s})\overline{\tilde{E}^+_{\mathbf{m}'}(\mathbf{s})} = |S_{\mathbf{m}}|\delta_{\mathbf{m}\mathbf{m}'}. \tag{57}$$

Proof. This proposition is proved in the same way as Proposition 1, but we have to take into account the difference between $\check{F}_M^n$ and $\hat{F}_M^n$. Due to the definition of the scalar product we have

$$\begin{aligned}\left\langle \tilde{E}^+_{\mathbf{m}}(\mathbf{s}), \tilde{E}^+_{\mathbf{m}'}(\mathbf{s})\right\rangle &= \sum_{\mathbf{s}\in F_N^n} \tilde{E}^+_{\mathbf{m}}(\mathbf{s})\overline{\tilde{E}^+_{\mathbf{m}'}(\mathbf{s})} \\ &= |S_n|^{-1}|S_{\mathbf{m}}| \sum_{w\in S_n}\prod_{i=1}^{n}\sum_{s_i\in F_N} e_{m_{w(i)}}(s_i)\overline{e_{m'_{w(i)}}(s_i)} = |S_{\mathbf{m}}|\delta_{\mathbf{m}\mathbf{m}'}.\end{aligned}$$

Here we have taken into account that there appear additional summands (with respect to (52)) because some summands on the right-hand side of (56) can coincide.

Since the functions $\tilde{E}^+_{\mathbf{m}}(\mathbf{s})$ are symmetric with respect to S_n, then

$$\sum_{\mathbf{s}\in F_N^n} \tilde{E}^+_{\mathbf{m}}(\mathbf{s})\overline{\tilde{E}^+_{\mathbf{m}'}(\mathbf{s})} = |S_n| \sum_{\mathbf{s}\in\check{F}_N^n} |S_{\mathbf{s}}|^{-1}\tilde{E}^+_{\mathbf{m}}(\mathbf{s})\overline{\tilde{E}^+_{\mathbf{m}'}(\mathbf{s})},$$

where we have taken into account that under an action by S_n upon $\check{F}_N^n$ a point $\mathbf{s}$ appears $|S_{\mathbf{s}}|$ times in F_N^n. This proves the proposition. □

Let f be a function on $\check{F}_N^n$ (or a symmetric function on F_N^n). Then it can be expanded in functions (56) as

$$f(\mathbf{s}) = \sum_{\mathbf{m}\in\check{D}_N^n} a_{\mathbf{m}}\tilde{E}^+_{\mathbf{m}}(\mathbf{s}). \tag{58}$$

The coefficients $a_{\mathbf{m}}$ are determined by the formula

$$a_{\mathbf{m}} = |S_n||S_{\mathbf{m}}|^{-1} \sum_{\mathbf{s}\in\check{F}_N^n} |S_{\mathbf{s}}|^{-1} f(\mathbf{s})\overline{\tilde{E}^+_{\mathbf{m}}(\mathbf{s})}. \tag{59}$$

Expansions (58) and (59) follow from the facts that numbers of elements in $\check{D}_N^n$ and $\check{F}_N^n$ are the same and from the orthogonality relation (57). We call expansions (58) and (59) the *symmetric multivariate discrete Fourier transforms*.

12. Eigenfunctions of (anti)symmetric Fourier transforms

Let $H_n(x), n = 0, 1, 2, \ldots,$ be the well-known Hermite polynomials of one variable. They satisfy the relation

$$\int_{-\infty}^{\infty} \mathrm{e}^{2\pi\,\mathrm{i}px}\,\mathrm{e}^{-\pi p^2} H_m(\sqrt{2\pi}\,p)dp = \mathrm{i}^{-m}\,\mathrm{e}^{-\pi x^2} H_m(\sqrt{2\pi}\,x) \tag{60}$$

(see, for example, subsection 12.2.4 in [17]).

We create polynomials of many variables

$$H_{\mathbf{m}}(\mathbf{x}) \equiv H_{m_1,m_2,\ldots,m_n}(x_1, x_2, \ldots, x_n) := H_{m_1}(x_1)H_{m_2}(x_2)\cdots H_{m_n}(x_n). \tag{61}$$

The functions

$$\mathrm{e}^{-|\mathbf{x}|^2/2} H_{\mathbf{m}}(\mathbf{x}), \qquad m_i = 0, 1, 2, \ldots, \qquad i = 1, 2, \ldots, n, \tag{62}$$

where $|\mathbf{x}|$ is the length of the vector $\mathbf{x}$, form an orthogonal basis of the Hilbert space $L^2(\mathbb{R}^n)$ with the scalar product $\langle f_1, f_2\rangle := \int_{\mathbb{R}^n} f_1(\mathbf{x})\overline{f_2(\mathbf{x})}\,\mathrm{d}\mathbf{x}$, where $\mathrm{d}\mathbf{x} = \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n$.

We make symmetrization and antisymmetrization of the functions

$$\mathcal{H}_{\mathbf{m}}(\mathbf{x}) := \mathrm{e}^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}(\sqrt{2\pi}\,\mathbf{x})$$

(obtained from (62) by replacing $\mathbf{x}$ by $\sqrt{2\pi}\,\mathbf{x}$) by means of the symmetric and antisymmetric multivariate exponential functions:

$$\int_{\mathbb{R}^n} E^+_\lambda(\mathbf{x})\,\mathrm{e}^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}(\sqrt{2\pi}\,\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathrm{i}^{-|\mathbf{m}|}\,\mathrm{e}^{-\pi|\lambda|^2} H^{\mathrm{sym}}_{\mathbf{m}}(\sqrt{2\pi}\,\lambda), \tag{63}$$

$$\int_{\mathbb{R}^n} E^-_\lambda(\mathbf{x})\,\mathrm{e}^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}(\sqrt{2\pi}\,\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathrm{i}^{-|\mathbf{m}|}\,\mathrm{e}^{-\pi|\lambda|^2} H^{\mathrm{anti}}_{\mathbf{m}}(\sqrt{2\pi}\,\lambda). \tag{64}$$

It is easy to see that the polynomials $H^{\mathrm{sym}}_{\mathbf{m}}$ and $H^{\mathrm{anti}}_{\mathbf{m}}$ indeed are symmetric and antisymmetric, respectively, with respect to the group S_n,

$$H^{\mathrm{sym}}_{\mathbf{m}}(w\lambda) = H^{\mathrm{sym}}_{\mathbf{m}}(\lambda), \qquad H^{\mathrm{anti}}_{\mathbf{m}}(w\lambda) = (\det w)H^{\mathrm{anti}}_{\mathbf{m}}(\lambda), \qquad w \in S_n.$$

For this reason, we may consider $H^{\mathrm{sym}}_{\mathbf{m}}(\lambda)$ for values of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $H^{\mathrm{anti}}_{\mathbf{m}}(\lambda)$ for values of λ, such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. The polynomials $H^{\mathrm{sym}}_{\mathbf{m}}$ are of the form

$$H^{\mathrm{sym}}_{\mathbf{m}}(\lambda) = \det{}^+(H_{m_i}(\lambda_j))^n_{i,j=1} \tag{65}$$

and the polynomials $H^{\mathrm{anti}}_{\mathbf{m}}$ of the form

$$H^{\mathrm{anti}}_{\mathbf{m}}(\lambda) = \det(H_{m_i}(\lambda_j))^n_{i,j=1}. \tag{66}$$

Moreover, $H^{\mathrm{anti}}_{\mathbf{m}}(\lambda) = 0$ if $m_i = m_{i+1}$ for some $i = 1, 2, \ldots, n-1$. For this reason, we may consider the polynomials $H^{\mathrm{sym}}_{\mathbf{m}}(\lambda)$ for integer n-tuples $\mathbf{m}$, such that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$ and the polynomials $H^{\mathrm{anti}}_{\mathbf{m}}(\lambda)$ for integer n-tuples $\mathbf{m}$, such that $m_1 > m_2 > \cdots > m_n$.

Let us apply the symmetric Fourier transform (42) (we denote it as $\mathfrak{F}$) to the symmetric functions (65). Taking into account formula (63) we obtain

$$\begin{aligned}\mathfrak{F}\big(\mathrm{e}^{-\pi|\mathbf{x}|^2} H^{\mathrm{sym}}_{\mathbf{m}}(\sqrt{2\pi}\,\mathbf{x})\big) &= \frac{1}{|S_n|}\int_{\mathbb{R}^n} E^+_\lambda(\mathbf{x})\,\mathrm{e}^{-\pi|\mathbf{x}|^2} H^{\mathrm{sym}}_{\mathbf{m}}(\sqrt{2\pi}\,\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &= \mathrm{i}^{-|\mathbf{m}|}\,\mathrm{e}^{-\pi|\lambda|^2} H^{\mathrm{sym}}_{\mathbf{m}}(\sqrt{2\pi}\,\lambda),\end{aligned}$$

that is, *the functions* $\mathrm{e}^{-\pi|\mathbf{x}|^2} H^{\mathrm{sym}}_{\mathbf{m}}(\sqrt{2\pi}\,\mathbf{x})$ *are eigenfunctions of the symmetric Fourier transform* $\mathfrak{F}$. Since these functions for $m_i = 0, 1, 2, \ldots, i = 1, 2, \ldots, n, m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$, form an orthogonal basis of the Hilbert space $L_0^2(\mathbb{R}^n)$ of functions from $L^2(\mathbb{R}^n)$ symmetric with respect to S_n (that is, of the Hilbert space $L^2(D_+)$), then they constitute a complete set of eigenfunctions of this transform. Thus, this transform has only four eigenvalues i, −i, 1, −1 in $L_0^2(\mathbb{R}^n)$. This means that we have $\mathfrak{F}^4 = 1$.

Now we apply the antisymmetric Fourier transform (40) (we denote it as $\tilde{\mathfrak{F}}$) to the antisymmetric function $\mathrm{e}^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}^{\mathrm{anti}}(\sqrt{2\pi}\mathbf{x})$. Taking into account formula (64) we obtain

$$\begin{aligned}\tilde{\mathfrak{F}}\big(\mathrm{e}^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}^{\mathrm{anti}}(\sqrt{2\pi}\mathbf{x})\big) &= \frac{1}{|S_n|}\int_{\mathbb{R}^n} E_\lambda^-(\mathbf{x})\,\mathrm{e}^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}^{\mathrm{anti}}(\sqrt{2\pi}\mathbf{x})\,\mathrm{d}\mathbf{x}\\ &= \mathrm{i}^{-|\mathbf{m}|}\,\mathrm{e}^{-\pi|\lambda|^2} H_{\mathbf{m}}^{\mathrm{anti}}(\sqrt{2\pi}\lambda),\end{aligned}$$

that is, *the functions* $\mathrm{e}^{-\pi|\mathbf{x}|^2} H_{\mathbf{m}}^{\mathrm{anti}}(\sqrt{2\pi}\mathbf{x})$ *are eigenfunctions of the transform* $\tilde{\mathfrak{F}}$. Since these functions for $m_i = 0, 1, 2, \ldots;\ i = 1, 2, \ldots, n, m_1 > m_2 > \cdots > m_n \geqslant 0$, form an orthogonal basis of the Hilbert space $L_-^2(\mathbb{R}^n)$ of functions from $L^2(\mathbb{R}^n)$ antisymmetric with respect to W, then they constitute a complete set of eigenfunctions of this transform. Thus, this transform has only four eigenvalues i, −i, 1, −1. This means that, as in the previous case, we have $\tilde{\mathfrak{F}}^4 = 1$.

It is possible to find eigenfunctions of the symmetric and antisymmetric finite Fourier transforms. It is done by using the results in [18, 19]. The method and results will be exposed in one of the forthcoming papers on this subject.

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